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A generalisation of the Runge–Lenz constant of classical motion in a central potential

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Abstract. A perihelion (unit) vector—the generalisation of the Runge-Lenz vector obtained by Fradkin is investigated. Its evolution, viewed as its dependence on the position and momentum vectors of a particle moving along its trajectory, demonstrates that, in general, it depends on time. For trajectories having the form of closed orbits with one pair of turning points, the perihelion vector turns out to be time independent and therefore a true integral of the motion. When a closed orbit possesses n pairs of turning points, an n-arm star of n different perihelion vectors is an invariant of the motion. In this case integrals of motion in the form of an n-rank tensor can be constructed from the perihelion vectors. Conditions for the existence of closed orbits are derived and illustrated by examples of the motion in the Kepler potential perturbed by a centrifugal-like term and in the isotropic harmonic oscillator potential similarly perturbed.

1. Introduction

The density functional theory of closed-shell atoms leads to a characterisation of the ground-state electron density $n(r)$ by a central potential energy $V(r)$. Unfortunately, this potential involves a contribution $\delta E_{xc}/\delta n(r)$ from an, as yet unknown, exchange and correlation energy functional $E_{xc}[n]$ (see, for example, Slater (1951) whose work was formally completed by Kohn and Sham (1965)).

Nevertheless, it is of interest to enquire whether the very existence of such a central potential $V(r)$ has consequences for atomic theory. Thus, it is known that, for motion in a bare Coulomb field, the classical Runge-Lenz vector of the Kepler problem has a ready generalisation to quantum mechanics (see Hostler 1967, Blinder 1984). (For the history of the classical Runge-Lenz vector, Goldstein (1975, 1976) may be consulted.)

This has prompted us to study further the classical problem of a suitable generalisation of the Runge-Lenz constant of motion for a bare Coulomb potential to an arbitrary central potential $V(r)$. However, there is already extensive relevant background to this study, which it is important first to summarise below.

Let us start by recording certain well established facts.

(a) For the Kepler (Coulomb) problem, with potential $V_K = -\lambda/r$, there exist constants (integrals) of the motion, which are the three components of the Runge-Lenz vector:

$$\boldsymbol{A} = (-2E)^{-1/2}[\tfrac{1}{2}(\boldsymbol{p}\times\boldsymbol{L} - \boldsymbol{L}\times\boldsymbol{p}) - \lambda m\boldsymbol{r}/r]. \tag{1.1}$$

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These components, together with the components of the orbital angular momentum vector $\boldsymbol{L}$, generate an algebra of the Lie group O_4 (see, for example, Fradkin 1967).

(*b*) For the isotropic harmonic oscillator (IHO) with potential $V_O = \mu^2 r^2/2m$, there exist constants of motion in the form of a symmetrical second-rank tensor (five independent components)

$$\mathscr{I}_{\alpha\beta} = \mu r_\alpha r_\beta + \frac{1}{\mu} p_\alpha p_\beta \tag{1.2}$$

which, together with L_α, generate an algebra of the Lie group SU_3 (see, for example, Fradkin 1965).

(*c*) For both potentials, in the case of classical (rather than quantum) mechanics, these Lie algebras are given in terms of the Poisson brackets (in quantum mechanics: commutators). This additional symmetry (compared with three-dimensional rotational symmetry: O_3) is termed dynamical symmetry. Extra degeneracy of quantal solutions results from these dynamical symmetries. It is to be noted that the classical and quantal cases have the same additional invariants (additional symmetry).

The question as to whether other rotationally invariant time-independent potentials have dynamical algebras for quantum systems has been investigated by Truax (1980). He demonstrates that oscillator, Coulomb and constant potentials are special, in the sense that their dengeneracy algebras are all larger than O_3 (i.e. SU_3, O_4, SU_3). All other central potentials possess O_3 symmetry only.

In the context of the present classical study, it is important to note that the same question has been addressed by numerous authors (Konar *et al* 1966, Fradkin 1967, Mukunda 1967, Stehle and Han 1967, Heintz 1974, Buch and Denman 1975a, b, Peres 1979, who was, however, not aware of Fradkin's (1967) paper especially). Without attempting a review at length of this extensive body of work, it should be noted that Peres rediscovered Fradkin's proposed generalisation of the Runge–Lenz vector, although the coefficients in the definition (scalar functions) are given by Peres via differential equations, which remain unsolved. He finds that it is not possible to derive a solution regular at both turning points.

What emerged from our study of the above papers on the classical Runge–Lenz problem was that a diversity of opinion still existed, in spite of the substantial body of work listed above. The main divergence of opinion centres around the existence of an additional symmetry (specifically O_4 and SU_3) for various potentials. Some workers connected this symmetry property with the generation of closed-orbit trajectories by a given force field. However, a different opinion was expressed, i.e. that such dynamical symmetry is quite a common property, possessed not only by central potentials, but which can even exist for arbitrary three-dimensional potentials. The motivation for the present work was to clarify this diversity of viewpoint.

Finally, in this brief review of earlier work, it should be mentioned that an approximate solution of the quantal motion for a general central potential has been attempted (Fivel 1966, Serebrennikov and Shabad 1971), the latter basing their work directly on Fradkin's generalisation of the Runge–Lenz vector.

To conclude this introduction, the motivation for the present study needs somewhat further emphasis. First, Fradkin himself, and others, were aware of the fact that the generalised Runge–Lenz vector (denoted as RV below) he proposed is only piecewise conserved, or alternatively that it is a multivalued constant. Nevertheless, these authors remained convinced that the RV proposed is a true integral of the motion because, as verified by Fradkin, the Poisson bracket of this vector with the Hamiltonian is zero.

Since, in our view, the above two facts cannot be reconciled, we have investigated this problem in a different manner, in order to resolve this contradiction. While Fradkin's proof of the constancy was of differential character, our approach may be termed an integral one, since we have studied the time evolution of Fradkin's RV via its dependence on $\boldsymbol{r}(t)$ and $\boldsymbol{p}(t)$, which are both obtained by an integration of the equation of motion.

In connection with the present investigation we find it necessary to generalise the notion of an integral (or constant) of the motion. The object, which can be defined solely in terms of the canonical variables ($\boldsymbol{r}$ and $\boldsymbol{p}$) of the moving particle and which is independent of time, will be called an invariant of the motion. We will apply this definition to an n-arm star of vectors, as well as to scalars, vectors or tensors (integrals of the motion).

The outline of the present paper is therefore as follows. In section 2, the time dependence of Fradkin's RV will be studied in some detail; for reasons set out there his RV will be subsequently termed the perihelion (unit) vector. Section 3 then consists of a formulation of the condition for a perihelion vector to be an integral of the motion, while in section 4 an invariant of the motion is constructed in the form of a star of perihelion vectors. Section 5 finally builds tensor integrals of the motion, constructed out of perihelion vectors. In the appendix, the time dependence of $\boldsymbol{r}$ and $\boldsymbol{p}$, describing the motion of a particle in an arbitrary central field, is set out.

2. Time dependence of Fradkin's vector $\hat{\boldsymbol{k}}$: the perihelion vector

Fradkin (1967) proposed the vector

$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p}) = \left(f - u\frac{\partial f}{\partial u}\right)\hat{\boldsymbol{r}} + L^{-2}\frac{\partial f}{\partial u}\boldsymbol{p}\times\boldsymbol{L} \tag{2.1}$$

which he termed a unit Runge vector, as a generalisation of the Runge-Lenz vector, known for the Kepler problem, to an arbitrary central potential $V(\boldsymbol{r}) = V(|\boldsymbol{r}|)$. Here the angular momentum

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} \tag{2.2}$$

is, of course, a constant of motion. The quantity

$$u = 1/r \tag{2.3}$$

will be used as an independent variable of the function f. Fradkin defined this function as

$$f = f(u;\, L^2, \mathscr{E}) = \cos\left(\frac{L}{m}\int_{u_0}^{u} \mathrm{d}u'\,[c^2 h(u')]^{-1/2}\right) \tag{2.4a}$$

for, in general, relativistic motion of the particle. Since we are interested in the non-relativistic case only, we shall rewrite (2.4a) by taking the limit $c\to\infty$ to find

$$f = f(u;\, L^2, E) = \cos\left(\frac{L}{m}\int_{u_0}^{u} \mathrm{d}u'[\tilde{h}(u')]^{-1/2}\right) \tag{2.4b}$$

where

$$\tilde{h}(u) = \lim_{c\to\infty}[c^2 h(u)] = \frac{2}{m}\left[E - V\left(\frac{1}{u}\right)\right] - \left(\frac{Lu}{m}\right)^2 \tag{2.5}$$

and the total non-relativistic energy E, another constant of motion, is given by

$$E = \lim_{c \to \infty} (\mathscr{E} - mc^2) = \frac{p^2}{2m} + V(r). \tag{2.6}$$

Throughout, $\hat{\boldsymbol{x}} = \boldsymbol{x}/x$ denotes a unit vector.

Using the identity

$$\boldsymbol{p} \times \boldsymbol{L} = \frac{\boldsymbol{r}L^2}{r^2} - \frac{\boldsymbol{L} \times \boldsymbol{r}(\boldsymbol{p} \cdot \boldsymbol{r})}{r^2 L^2} \tag{2.7}$$

which follows from the definition (2.2), equation (2.1) can be transformed to

$$\hat{\boldsymbol{k}} = f\hat{\boldsymbol{r}} - \frac{(\boldsymbol{p} \cdot \boldsymbol{r})}{Lr} \frac{\partial f}{\partial u} \hat{\boldsymbol{L}} \times \hat{\boldsymbol{r}}. \tag{2.8}$$

The immediate aim below is to study the time dependence of this Fradkin vector $\hat{\boldsymbol{k}}$. This time dependence stems from that of the canonical variables $\boldsymbol{r} = \boldsymbol{r}(t)$ and $\boldsymbol{p} = \boldsymbol{p}(t)$, corresponding to the particle motion in its orbit (see the appendix for details). To facilitate this study, let us choose the z axis along the vector $\boldsymbol{L}$. Then one can write $\hat{\boldsymbol{L}} = \{0, 0, 1\}$ and using the polar coordinates r, ϕ:

$$\boldsymbol{r} = r\boldsymbol{e}(\phi) \tag{2.9}$$

$$\boldsymbol{e}(\phi) = \{\cos\phi, \sin\phi, 0\} = \hat{\boldsymbol{r}}. \tag{2.10}$$

Thus

$$\boldsymbol{p} = m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = m\dot{\boldsymbol{r}} = m[\dot{r}\boldsymbol{e}(\phi) + r\dot{\phi}\boldsymbol{e}'(\phi)] \tag{2.11}$$

where

$$\begin{aligned} \boldsymbol{e}'(\phi) &= \frac{\mathrm{d}}{\mathrm{d}\phi}\boldsymbol{e}(\phi) = \{-\sin\phi, \cos\phi, 0\} = \hat{\boldsymbol{L}} \times \hat{\boldsymbol{r}} \\ \boldsymbol{e}'(\phi) \cdot \boldsymbol{e}(\phi) &= 0. \end{aligned} \tag{2.12}$$

According to (2.2), one can write

$$\boldsymbol{L} = mr\boldsymbol{e} \times [\dot{r}\boldsymbol{e} + r\dot{\phi}\boldsymbol{e}'] = mr^2\dot{\phi}\boldsymbol{e} \times \boldsymbol{e}' = mr^2\dot{\phi}\{0, 0, 1\}. \tag{2.13}$$

From (2.10) and (2.12), it follows that (2.8) can be expressed in the form

$$\hat{\boldsymbol{k}} = f\hat{\boldsymbol{r}} - g\hat{\boldsymbol{L}} \times \hat{\boldsymbol{r}} = f\boldsymbol{e}(\phi) - g\boldsymbol{e}'(\phi) \tag{2.14}$$

where

$$g = \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{Lr}\frac{\partial f}{\partial u}. \tag{2.15}$$

Comparing (2.5) with (A1), it follows that $\tilde{h}(u) = \dot{r}^2(1/u)$. In order to impose on Fradkin's general solution (2.4*b*) our intial conditions set out in the appendix (above (A3)), his arbitrary constant u_0 is chosen to be

$$u_0 = 1/r_p \tag{2.16}$$

(r_p is defined in connection with (A2)). We also resolve the ambiguity of sign associated with $\tilde{h}^{1/2}$, choosing the plus sign, to find (compare (A4))

$$\tilde{h}^{1/2}(u) = \dot{r}_1(1/u). \tag{2.17}$$

Therefore (2.4b) can be rewritten in terms of ϕ_1 in (A3), with $r' = 1/u'$, as

$$f = \cos\left(\frac{L}{m}\int_{r_p}^{1/u} -\frac{\mathrm{d}r'}{r'^2\dot{r}_1(r')}\right) = \cos(-\phi_1(1/u)) = \cos(\phi_1(r)). \tag{2.18}$$

By differentiation (keeping $r = 1/u$ (2.3)):

$$\frac{\partial f}{\partial u} = -r^2\frac{\partial f}{\partial r} = \sin(\phi_1(r))\frac{L}{m\dot{r}_1(r)}. \tag{2.19}$$

From (2.9)-(2.12) we have

$$\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{Lr} = \frac{mr\boldsymbol{e}\cdot(\dot{r}\boldsymbol{e} + r\dot{\phi}\boldsymbol{e}')}{Lr} = \frac{m\dot{r}}{L}. \tag{2.20}$$

Thus the function g in (2.15) becomes

$$g = \frac{\dot{r}}{\dot{r}_1(r)}\sin(\phi_1(r)) = \mathrm{sgn}(\dot{r})\sin(\phi_1(r)) \tag{2.21}$$

where the second step follows from (A22a). It can then be seen that the functions f and g in (2.18) and (2.21) can conveniently be written in terms of a single function Φ as

$$f = \cos\Phi \qquad g = \sin\Phi \tag{2.22}$$

where

$$\Phi = \Phi(\boldsymbol{r}, \boldsymbol{p}) = \mathrm{sgn}(\boldsymbol{r}\cdot\boldsymbol{p})\phi_1(r). \tag{2.23}$$

Thus the Fradkin vector $\hat{\boldsymbol{k}}$ in (2.14), by means of (2.22), (2.10) and (2.12), has components

$$\hat{\boldsymbol{k}} = \{\cos\Phi\cos\phi + \sin\Phi\sin\phi, \cos\Phi\sin\phi - \sin\Phi\cos\phi, 0\}. \tag{2.24}$$

At this point, let us define a vector function

$$\boldsymbol{e}(\theta) = \{\cos\theta, \sin\theta, 0\} \tag{2.25}$$

which is a generalisation of the definition (2.10). Here θ is now an arbitrary variable, whereas ϕ denotes the azimuthal coordinate of the particle position. The vector $\hat{\boldsymbol{k}}$ in (2.24) has an especially simple form in terms of $\boldsymbol{e}$ (θ):

$$\hat{\boldsymbol{k}} = \boldsymbol{e}(\phi - \Phi(\boldsymbol{r}, \boldsymbol{p})) \tag{2.26}$$

which is suitable for investigating its time dependence. Due to (2.20), the time dependence of Φ in (2.23) is

$$\Phi(t) = \Phi(\boldsymbol{r}(t), \boldsymbol{p}(t)) = \mathrm{sgn}(\dot{r}(t))\phi_1(r(t)) \tag{2.27}$$

From (A18), (A22b) and (A16) we find the relation

$$\Phi(\tau) = \phi(\tau) \qquad \text{for } \tau \in (-T/2, T/2) \tag{2.28}$$

and from (A11) and (A19)

$$\Phi(NT + \tau) = \Phi(\tau) \qquad N \text{ an integer}. \tag{2.29}$$

Therefore, after inserting (A17), (2.29) and (2.28) into (2.26), we finally obtain $\hat{k}$ at arbitrary time $t = NT + \tau$ as

$$\hat{k}(NT+\tau) = e(N\Delta\phi + \phi(\tau) - \Phi(\tau)) = e(N\Delta\phi) \tag{2.30}$$

for

$$\tau \in (-T/2, T/2) \qquad N = 0, \pm 1, \pm 2, \ldots .$$

It can be seen $\hat{k}(t)$ is constant for 'almost all time', except at the specific times $t = (N+\frac{1}{2})T$, with N integral, i.e. when the particle passes some aphelion, at which the direction of $\hat{k}$ 'jumps' by the angle $\Delta\phi$ (given by (A10) and (A3)) to the direction pointing to the next perihelion at a later time. It should be noted that, if the initial condition were chosen at an aphelion (instead of at a perihelion), the role of aphelion and perihelion in the above conclusion would be reversed.

The result (2.30) can be alternatively described by saying that $\hat{k}$ is a multivalued function (i.e. having many branches numbered by N) which is constant in time. In such a formulation, the condition for a quantity to be an integral of motion is not only that it shall be constant, but also that it should be a single-valued function of the canonical variables, as set out, for example, by Landau and Lifshitz (1976).

Fradkin (1967) was indeed aware of the possibility that $\hat{k}$ may be multivalued, but he considered it inessential. The definition of conserved quantities that he adopted was that such quantities (not explicitly functions of time) should have zero Poisson brackets with the energy. Thus, when he explicitly verified that

$$\{E(\boldsymbol{r}, \boldsymbol{p}), \hat{k}(\boldsymbol{r}, \boldsymbol{p})\}_{\text{Poisson bracket}} = 0 \tag{2.31}$$

which is, of course, equivalent to $d\hat{k}(t)/dt = 0$, he considered $\hat{k}$ to be an integral of the motion.

Unfortunately, as was pointed out by Serebrennikov and Shabad (1971) and also by Peres (1979), the scalar coefficient functions, occurring in the definition (2.1), are singular at the turning points of the orbit. Therefore, in fact, Fradkin's result (2.31) does not determine $d\hat{k}/dt$ there. Indeed, we know from (2.30) that, just at one of these points, an aphelion, $\hat{k}$ changes direction aburptly (this happens at a perihelion with another set of initial conditions).

The conclusion therefore is that the time dependence of $\hat{k}$ established in (2.30) demonstrates that $\hat{k}$ is not an integral of the motion for an arbitrary central potential† and an arbitrary orbit (i.e. arbitrary E and L).

Since it has been established in the present work that $\hat{k}$ is always directed towards a perihelion, it will be called the perihelion vector in the following.

3. Condition for a perihelion vector to be an integral of the motion

An obvious condition for the perihelion vector $\hat{k}(\boldsymbol{r}, \boldsymbol{p})$ to be an integral of the motion, i.e. for $\hat{k}(r) = \hat{k}(\boldsymbol{r}(t), \boldsymbol{p}(t)$ to be constant in time, follows from (2.30). It is that

$$\Delta\phi = \Delta\phi_V(E, L) = 2\pi l \qquad l = 1, 2, 3, \ldots \tag{3.1}$$

† Fradkin, and other authors, give a proof that k satisfies the conditions of an integral of the motion 'almost everywhere'.

where $\Delta\phi$, defined by (A10), (A3) and (A4) for a given potential $V(r)$, depends on the orbit parameters E and L. For a general potential, the condition (3.1) may be met for some particular values of E and L only, i.e. for some restricted class of orbits.

In the Kepler (Coulomb) case $V(r) = V_K(r) = -\lambda/r$, $\lambda > 0$, we have

$$\Delta\phi_K = 2\int_{r_p}^{r_a} \frac{\mathrm{d}r}{r}\left(\frac{r_p r_a}{(r-r_p)(r_a-r)}\right)^{1/2} \tag{3.2}$$

where r_p and r_a are functions of E and L which are readily obtained from (A2) provided that

$$\frac{-m\lambda^2}{2L^2} < E < 0. \tag{3.3}$$

By contour integration in the complex r plane one obtains

$$\Delta\phi_K = 2\pi \tag{3.4}$$

and it follows that $\Delta\phi_K$ in the bare Coulomb case does not depend on the orbit parameters E and L nor on the potential strength λ. This property of $\Delta\phi_K$, as pointed out by Konar *et al* (1966), admits the existence of a dynamical group (O_4) for motion in the Coulomb field.

3.1. Example of Kepler potential perturbed by centrifugal potential energy

As an example of a central potential lacking the above properties, let us next consider a Kepler potential perturbed by a centrifugal-like term $V_C(r) = -\Lambda/2mr^2$:

$$V_{KC}(r) = V_K(r) + V_C(r) = -\lambda/r - \Lambda/2mr^2. \tag{3.5}$$

This leads to the radial velocity (see (A4))

$$\dot{r}_1(r) = \left(\frac{2}{m}[E - V_K(r)] - \frac{L_{\mathrm{eff}}^2}{m^2r^2}\right)^{1/2} \tag{3.6}$$

which differs from the corresponding Kepler quantity by the replacement of L^2 by

$$L_{\mathrm{eff}}^2 = L^2 - \Lambda \tag{3.7}$$

which also modifies the conditions (3.3) to

$$\Lambda < L^2 \qquad -\frac{m\lambda^2}{2L_{\mathrm{eff}}^2} < E < 0. \tag{3.8}$$

After inserting (3.6) into (A3) and (A10) we obtain

$$\Delta\phi_{KC} = \frac{2L}{L_{\mathrm{eff}}}\int_{r_p}^{r_a} \frac{\mathrm{d}r}{r}\left(\frac{r_p r_a}{(r-r_p)(r_a-r)}\right)^{1/2}. \tag{3.9}$$

This result is very similar to (3.2), although r_p and r_a occurring in (3.9) are different. But the result after integration is independent of r_p and r_a (see (3.4) and (3.2)) so that

$$\Delta\phi_{KC} = 2\pi(1 - \Lambda/L^2)^{-1/2}. \tag{3.10}$$

Therefore the perihelion vector $\hat{k}$ may be an integral of the motion in the field V_{KC} (i.e (3.1) may be satisfied) for the orbits having angular momentum

$$L = L_{KC}(l) = [\Lambda/(1 - l^{-2})]^{1/2} \qquad l = 2, 3, \ldots \tag{3.11}$$

provided that the strength Λ of the V_C term is positive. A further example will be discussed in section 5.

4. Invariant of the motion in form of a star of perihelion vectors

It is evident that orbits satisfying the condition (3.1) form a closed trajectory. In the literature, the existence of the O_4 symmetry in the Kepler problem and the SU_3 symmetry in the isotropic harmonic oscillator (IHO) is often connected with the fact that both of these examples have orbits that are closed. Let us investigate therefore a more general condition than (3.1) for an orbit to be closed, namely

$$\Delta\phi_V(E, L) = 2\pi l/n \qquad l, n = 1, 2, 3, \ldots \tag{4.1}$$

with l/n being a non-reducible fraction. The condition (3.1) is evidently the particular case of (4.1) corresponding to $n = 1$. Satisfying (4.1) means that during l rotations around the potential centre, a particle approaches perihelions (and also aphelions) n times. Here the period of the motion (the time to 'close' the trajectory) is nT. Following Konar *et al* (1966), the orbits satisfying (4.1) will be referred to as orbits with multiplicity n.

For the treatment of such orbits, it is convenient to use the notation (compare the definition (2.25))

$$\boldsymbol{e}^{n,j} = \boldsymbol{e}(2\pi j/n) \tag{4.2}$$

and also to employ the notion of correspondence ($\doteq$) between a vector lying in the plane of motion and a complex number:

$$\{x, y, 0\} \doteq x + \mathrm{i}y. \tag{4.3}$$

Thus the vector $\boldsymbol{e}^{n,j}$ in (4.2) can be written as

$$\boldsymbol{e}^{n,j} = \{\cos(2\pi j/n), \sin(2\pi j/n), 0\} \doteq [\exp(\mathrm{i}\, 2\pi/n)]^j. \tag{4.4}$$

As is well known, a set of complex numbers

$$\{C_n^l,\ l = 0, 1, \ldots, (n-1);\ C_n^1 = \exp(\mathrm{i}\, 2\pi/n)\} \tag{4.5}$$

forms an Abelian cyclic group of order n with respect to multiplication. Thus C_n^N for an arbitrary integer N is a member of this set, while the sets $\{C_n^{lj},\ l/n$ non-reducible, $j = 0, 1, \ldots, (n-1)\}$ and $\{C_n^{N+l},\ l = 0, 1, \ldots, (n-1)\}$ are identical to the set (4.5).

Now, for the orbit of multiplicity n, let us find the perihelion vector $\hat{\boldsymbol{k}}$ at some arbitrary time written as $t = \tau + (Mn + j)T$ with M integral; $j = 0, 1, \ldots, n-1$; $\tau \in (-T/2, T/2)$. Inserting (4.1) into (2.30), we obtain for the perihelion vector at this time

$$\begin{aligned} \hat{\boldsymbol{k}}(\tau + (Mn + j)T) &= \boldsymbol{e}((Mn + j)2\pi l/n) \\ &= \boldsymbol{e}^{n,lj} \doteq C_n^{lj}. \end{aligned} \tag{4.6}$$

Due to the above-mentioned group property, it can be seen that, during the particle motion its perihelion vector $\hat{\boldsymbol{k}}$ attains only a finite number n of distinct directions: $\boldsymbol{e}^{n,q}$, $q = 0, 1, \ldots, (n-1)$, which, of course, are pointing towards the n perihelions of the closed trajectory under consideration. Thus a set of n vectors defined as

$$\hat{\boldsymbol{k}}^{n,q}(\boldsymbol{r}, \boldsymbol{p}) = \cos(2\pi q/n)\hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p}) + \sin(2\pi q/n)\hat{\boldsymbol{L}} \times \hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p}) \qquad q = 0, 1, \ldots, (n-1) \tag{4.7}$$

should represent the desired invariant of motion for a closed orbit with multiplicity n. At $t = 0$, this set consists of the original perihelion vector $\hat{\boldsymbol{k}}^{n,0} = \hat{\boldsymbol{k}}(0) = \boldsymbol{e}^{n,0}$ and the remaining $(n-1)$ perihelion vectors $\boldsymbol{e}^{n,q}$, $q = 1, 2, \ldots, (n-1)$. The time dependence of the qth vector of the set (4.7), due to the known time dependence of $\hat{k}(\boldsymbol{r}, \boldsymbol{p})$ in (2.26), is

$$\hat{k}^{n,q}(t) \equiv \hat{k}^{n,q}(\boldsymbol{r}(t), \boldsymbol{r}(t)) = e(2\pi q/n + \phi(t) - \Phi(t)) \tag{4.8}$$

and thus, from (2.28), (2.29) and (A17), it follows by analogy, as was seen in (4.6), that

$$\hat{k}^{n,q}(\tau + (Mn+j)T) = e(2\pi(q+lj)/n) = \hat{k}^{n,lj+q}(0) \doteq C_n^{lj+q}. \tag{4.9}$$

But, because of the group property of the set (4.5), the set $\{C_n^{lj+q}, q = 0, 1, \ldots, (n-1)\}$ is the same as $\{C_n^q, q = 0, 1, \ldots, (n-1)\}$. Therefore the set (4.7) does not depend on time: it is really an invariant of motion. According to its geometrical meaning, it will be referred to as an n-arm star of perihelion vectors.

If is worth noting here that the definition (4.7) is meaningful also for a non-integer q. Therefore a star consisting of n such vectors (with fixed parameter a)

$$\hat{k}^{n,a+q}(\boldsymbol{r}, \boldsymbol{p}) \qquad 0 \leq a < 1;\ q = 0, 1, \ldots, (n-1) \tag{4.10}$$

is also an invariant. It differs from the perihelion star (4.7) by a rotation around the potential centre by an angle $2\pi a/n$. In particular, a star with $a = \frac{1}{2}$ coincides with the aphelion star.

5. Tensor integrals of the motion constructed from perihelion vectors

Besides the 'geometrical' invariants: n-arm stars, discussed in section 4, one can construct from the vectors $\hat{k}^{n,a+q}(\boldsymbol{r}, \boldsymbol{p})$ other invariants having the form of tensors.

Let us introduce a notation $\boldsymbol{A} \wedge \boldsymbol{B}$ for a second-rank tensor which is a direct (Kronecker) product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$(\boldsymbol{A} \wedge \boldsymbol{B})_{\alpha\beta} = A_\alpha B_\beta. \tag{5.1}$$

Here the subscripts α, β label the indices of the Cartesian coordinates. This definition has an obvious extension to any number of factors.

Using the above notation, we next define the following tensor of rank n:

$$\mathcal{I}^{n,a}(\boldsymbol{r}, \boldsymbol{p}) = \sum_{q=0}^{n-1} \hat{k}^{n,a+q+1} \wedge \hat{k}^{n,a+q+2} \wedge \ldots \wedge \hat{k}^{n,a+q+n} \tag{5.2}$$

where each $\hat{k}^{n,b}$ means $\hat{k}^{n,b}(\boldsymbol{r}, \boldsymbol{p})$ (see (4.7)). The time dependence of this tensor is dictated by (4.9):

$$\mathcal{I}^{n,a}(\tau + (Mn+j)T) = \mathcal{I}^{n,a+lj}(0) = \mathcal{I}^{n,a}(0) \tag{5.3}$$

where the second equality is due to the cyclic property

$$\hat{k}^{n,b+n}(\boldsymbol{r}, \boldsymbol{p}) = \hat{k}^{n,b}(\boldsymbol{r}, \boldsymbol{p}) \tag{5.4}$$

(which follows from (4.7)) and due to the freedom to change the order of summed terms. Thus it has been established that the tensor $\mathcal{I}^{n,a}(\boldsymbol{r}, \boldsymbol{p})$ in (5.2) is an integral of motion for closed orbits of multiplicity n, i.e. satisfying (4.1).

A linear combination of such tensors with different values of a parameter a is, of course, also an integral of the motion:

$$\boldsymbol{\mathcal{I}}^{n}(\boldsymbol{r}, \boldsymbol{p}) = \sum_a C_a(E, L) \boldsymbol{\mathcal{I}}^{n,a}(\boldsymbol{r}, \boldsymbol{p}). \tag{5.5}$$

One can also imagine tensors of rank $2n$ (or $3n$, etc) constructed as a linear combination of terms like $\mathscr{I}^{n,a} \wedge \mathscr{I}^{n,b}$ (or $\mathscr{I}^{n,a} \wedge \mathscr{I}^{n,b} \wedge \mathscr{I}^{n,c}$, etc). However, the problem remains of how to choose sets of coefficients C_a, C_{ab}, $C_{abc}, \ldots$, to obtain only the independent invariants. Another related problem left open is which of the independent invariants 'commute' mutually (in the sense of Poisson brackets).

It has to be noted here that no invariants are obtained by contraction of the above-mentioned tensors with the invariant vector $\boldsymbol{L}$, because

$$\hat{\boldsymbol{k}}^{n,a} \cdot \boldsymbol{L} = 0 \tag{5.6}$$

this result following directly from (4.7) and (2.14).

5.1. *Examples of isotropic harmonic oscillator: with and without perturbation*

Now let us consider some examples. For the case of the IHO with the potential $V_{\mathrm{O}}(r) = \mu^2 r^2/2m$ we find

$$\Delta\phi_{\mathrm{O}} = 2 \int_{r_{\mathrm{p}}}^{r_{\mathrm{a}}} \frac{\mathrm{d}r}{r} \left(\frac{r_{\mathrm{p}}^2 r_{\mathrm{a}}^2}{(r^2 - r_{\mathrm{p}}^2)(r_{\mathrm{a}}^2 - r^2)} \right)^{1/2} \tag{5.7}$$

where r_p and r_a are the roots of (A2), provided that

$$E > |\mu| L/m. \tag{5.8}$$

Using a new variable $\rho = r^2$ we reduce the integral (5.7) to the known case (3.2), evaluated in (3.4):

$$\Delta\phi_{\mathrm{O}} = 2 \int_{r_{\mathrm{p}}^2}^{r_{\mathrm{a}}^2} \frac{\mathrm{d}\rho}{2\rho} \left(\frac{r_{\mathrm{p}}^2 r_{\mathrm{a}}^2}{(\rho - r_{\mathrm{p}}^2)(r_{\mathrm{a}}^2 - \rho)} \right)^{1/2} = 2\pi \tfrac{1}{2}. \tag{5.9}$$

Thus the IHO orbit has multiplicity 2, for any E and L satisfying (5.8).

The same multiplicity ($n = 2$) may also be obtained for the case of the perturbed Kepler potential V_{KC}, (3.5), having the L dependent $\Delta\phi = \Delta\phi_{\mathrm{KC}}$; (3.10). In order to have $n = 2$ and $l = 1$ in (4.1), the strength parameter Λ of the V_{C} term must be negative and the angular momentum of the motion must be

$$L = (-\Lambda/3)^{1/2} \tag{5.10}$$

while the case of $l = 3, 5, \ldots$, can be achieved for Λ positive and orbits having

$$L = L(l) = \{\Lambda/[1 - (2/l)^2]\}^{1/2}. \tag{5.11}$$

The invariants of the motion in the case of orbits of multiplicity $n = 2$ are especially simple. The two-arm star is just $\{\hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p}), -\hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p})\}$ and thus the second-rank tensor (5.2) is

$$\mathscr{I}^{2,a}_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{p}) = \hat{k}^{2,a}_{\alpha}(-\hat{k}^{2,a}_{\beta}) + (-\hat{k}^{2,a}_{\alpha})\hat{k}^{2,a}_{\beta} = -2\hat{k}^{2,a}_{\alpha}\hat{k}^{2,a}_{\beta}. \tag{5.12}$$

Choosing a set of a to consist of 0 and $\frac{1}{2}$ we have for (5.5)

$$\mathscr{I}_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{p}) = F(L^2, E)\hat{k}_{\alpha}(\boldsymbol{r}, \boldsymbol{p})\hat{k}_{\beta}(\boldsymbol{r}, \boldsymbol{p}) + G(L^2, E)(\hat{\boldsymbol{L}} \times \hat{\boldsymbol{k}})_{\alpha}(\hat{\boldsymbol{L}} \times \hat{\boldsymbol{k}})_{\beta} \tag{5.13}$$

which is a second-rank symmetric tensor. This is the same as proposed by Fradkin (1967) to construct a Lie algebra for the SU_3 group. With

$$F, G = mE/\mu \mp [(mE/\mu)^2 - L^2]^{1/2} \tag{5.14}$$

proposed by Fradkin (1967) for the IHO case, one obtains from (5.13) the well known invariant tensor

$$\mathcal{J}_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{p}) = \mu r_\alpha r_\beta + (1/\mu) p_\alpha p_\beta. \tag{5.15}$$

It should be pointed out that, since the result (5.9) for the IHO problem does not depend on the orbit parameters E and L, it follows that the additional symmetry SU_3 characterises any motion in this field. But, because a single perihelion vector $\hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p})$ is not an integral of the motion for the IHO problem, there is no symmetry O_4 connected withit. For all central potentials different from the Kepler or IHO forms, additional invariants (besides E and L) occur for special values of orbit parameters E and L only, as was discussed in, for example, (3.11), (5.10), (5.11) and (5.18).

The problem of Runge vectors for the IHO case was investigated in detail by Buch and Denman (1975b). Following Fradkin, these authors obtained an explicit expression for a unit Runge vector $\hat{\boldsymbol{k}}$, (2.1), for the IHO potential and they found that $\hat{\boldsymbol{k}}$ is not a constant of motion because it 'discontinuously reverses its direction when the particle crosses apogee'. This is in agreement with the general considerations outlined above and applied to orbits having multiplicity 2. But these authors were tempted to construct from $\boldsymbol{k}$ 'a true constant of the motion' by furnishing $\hat{\boldsymbol{k}}$ with an additional sign according to the prescription: 'the plus sign is used when the particle is on the right half of the orbit, and the minus sign when on the left half'. While such an object would indeed be constant in time, it cannot be called a constant of the motion, because it is not possible to formulate their prescription solely in terms of the canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$.

Having the IHO results (5.7)-(5.9), it is easy to find a solution for the case of an oscillator perturbed by a centrifugal-like potential V_C:

$$V_{OC}(t) = \mu^2 r^2/2m - \Lambda/2mr^2. \tag{5.16}$$

As in the perturbed Kepler case, (3.5), the effective angular momentum (3.7) occurs in the expression for the radial velocity, which leads to

$$\Delta\phi_{OC} = 2\pi\tfrac{1}{2}(1 - \Lambda/L^2)^{-1/2} \tag{5.17}$$

provided that $L^2 > \Lambda$, $E > (|\mu|/m)\ (L^2 - \Lambda)^{1/2}$. The orbit satisfying the condition (4.1) can be realised for the angular momentum values

$$L = L_{OC}(n, l) = \{\Lambda/[1 - (n/2l)^2]\}^{1/2}. \tag{5.18}$$

The case of $l > n/2$ can be acheived for potentials having Λ positive and the case of $l < n/2$ for Λ negative.

If we ask how the condition (4.1) can be satisfied for motion in the perturbed Kepler potential $V_{KC}(\mathrm{r})$, (3.5), we immediately obtain from (3.10) that the angular momentum values must be

$$L = \mathrm{L}_{KC}(n, l) = \{\Lambda/[1 - (n/l)^2]\}^{1/2} \tag{5.19}$$

where the case of $l > n$ can be achieved for potentials with Λ positive and the case $l < n$ for Λ negative.

6. Summary

The time dependence of the vector $\hat{\boldsymbol{k}}(\boldsymbol{r}, \boldsymbol{p})$ in (2.1), proposed by Fradkin (1967) as a generalisation of the Runge-Lenz vector, is fully investigated here. While Fradkin, with the same aim, evaluated the Poisson bracket and obtained the result (2.31), the approach adopted in the present work is to evaluate directly $\hat{k}(t) = \hat{k}(r(t), p(t))$. The explicit forms of $r(t)$ and $p(t)$ corresponding to the particle motion along its orbit are constructed in the appendix. These lead to the dependence of $\hat{k}$ on t exhibited in (2.30). This demonstrates therefore that, in general, $\hat{k}$ changes in time and is therefore not an integral of the motion.

Because of this situation, the necessary condition given in (3.1) has been formulated which will ensure that $\hat{k}(r, p)$ is an integral of the motion. It is verified that, for an arbitrary orbit in the Kepler field, this condition is fulfilled (see (3.4)) so that $\hat{k}$ is proportional to the Runge–Lenz vector. For other potentials, this condition can be met for some particular values of E and L pertaining to the orbit. This is illustrated by taking the example of the Kepler potential perturbed by a centrifugal term as in (3.5). In this case orbits for which L has values (3.11) and energy E bounded by (3.8) possesses an integral of the motion $\hat{k}(r, p)$.

Closed orbits satisfying the condition (4.1), i.e. having n perihelions and n aphelions, are investigated. It is shown that an n-arm star consisting of all n perihelion vectors, (4.7), and corresponding to it the nth-rank tensor, (5.5) with (5.2), are both invariants of the motion in this case, which is proved in (4.9) and (5.3). The question as to how many independent tensor invariants can be constructed via an appropriate choice of the coefficients $C_a(E, L)$ in (5.5) remains open. The isotropic harmonic oscillator fulfils condition (4.1) with $n = 2$, $l = 1$, according to (5.9), for arbitrary E and L, bounded by (5.8). The invariant star consists of $\hat{k}$ and $-\hat{k}$, while the second-rank tensor has a well known form given by (5.15). The perturbed oscillator (5.16), the other example considered in the present study, can possess orbits with any combination of n and l and therefore corresponding invariants: n-arm stars and nth-rank tensors, provided that the value of L is chosen according to (5.18). A similar situation is established for the perturbed Kepler potential (3.5) with the value of L given by (5.19).

Returning to the related question of dynamical symmetry mentioned in the introduction, the existence of an integral of motion having generalised Runge–Lenz character is uniquely related to the existence of closed orbits. While Coulomb and harmonic potentials have this property for arbitrary L and E, it has been demonstrated here that a general central potential gives rise to closed orbits only for particular values of L and E. Such closed orbits clearly have very definite geometrical symmetry which, however, is not to be associated with the particular central field alone.

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Appendix. Time dependence of canonical variables during a finite motion of a particle in a central potential.

Although in general it is a textbook problem (see, e.g., Landau and Lifshiftz, 1976), we find it necessary to discuss explicit formulae for $r(t)$ and $p(t)$ because, for one thing, we want to remove some ambiguity concerning the sign of the radial velocity appearing in the work of Buch and Benman (1975b).

The motion under consideration is described in terms of polar coordinates introduced in section 2, (2.9)-(2.12). The squared radial velocity can be obtained from (2.6), together with (2.11) and (2.13), as

$$\dot{r}^2(r)=\frac{2}{m}[E-V(r)]-\frac{L^2}{m^2r^2}. \tag{A1}$$

The roots r_p and r_a of the equation

$$\dot{r}^2(r)=0 \tag{A2}$$

ensuring that $\dot{r}^2(r)>0$, for $0<r_p<r<r_a$, characterise the turning points. They define the distance from the potential centre to a perihelion and an aphelion of the orbit, respectively.

Let us choose as the initial conditions $\phi=0$, $r=r_p$ and $\dot{\phi}>0$ at $t=0$, i.e. a particle passing a perihelion. The solution, valid for a time interval from zero to the moment when the particle reaches the aphelion, is known to be

$$\phi_1(r)=\frac{L}{m}\int_{r_p}^{r}\frac{\mathrm{d}r'}{r'^2\dot{r}_1(r')} \tag{A3}$$

where the sign of the expression for the radial velocity, obtained from (A1),

$$\dot{r}_1(r)=\left\{\frac{2}{m}[E-V(r)]-\frac{L^2}{m^2r^2}\right\}^{1/2}\geqslant 0 \tag{A4}$$

is chosen positive in order to be consistent with r and ϕ being increasing functions during the time interval considered. The corresponding time dependence of the motion is given by

$$t_1(r)=\int_{r_p}^{r}\frac{\mathrm{d}r'}{\dot{r}_1(r')}. \tag{A5}$$

Next, during the motion after passing the aphelion but before reaching the perihelion, r is diminishing, so the second radical of (A1) should be used:

$$\dot{r}_2(r)=-\dot{r}_1(r) \tag{A6}$$

and then branches of the ϕ and t, which are continuing the relations (A3) and (A5), can be obtained:

$$\begin{aligned}\phi_2(r)&=\phi_1(r_a)+\frac{L}{m}\int_{r_a}^{r}\frac{\mathrm{d}r'}{r'^2\dot{r}_2(r')}\\&=\phi_1(r_a)+\frac{L}{m}\left(\int_{r_p}^{r_a}-\int_{r_p}^{r}\right)\frac{\mathrm{d}r'}{r'^2\dot{r}_1(r')}\\&=2\phi_1(r_a)-\phi_1(r)\end{aligned} \tag{A7}$$

and

$$t_2(r)=t_1(r_a)+\frac{L}{m}\int_{r_a}^{r}\frac{\mathrm{d}r'}{\dot{r}_2(r')}=2t_1(r_a)-t_1(r). \tag{A8}$$

In an analogous fashion, subsequent branches of ϕ and t can be calculated for the later motion. We see that r oscillates back and forth between r_p and r_a with period

$$T = 2t_1(r_a) \tag{A9}$$

during which time the azimuthal coordinate ϕ increases by

$$\Delta\phi = 2\phi_1(r_a). \tag{A10}$$

Because $t_1(r)$ in (A5) is a monotonic function, there exists a function $r_1(t)$, reciprocal to $t_1(r)$, i.e. satisfying the equation $t_1(r_1(t)) = t$. In terms of this, the dependence of the first two branches of t on r, in (A5) and (A8), can be inverted to give r against t:

$$r(t) = \begin{cases} r_1(t) & \text{for } 0 \leqslant t \leqslant T/2 \\ r_1(T-t) & \text{for } T/2 \leqslant t \leqslant T. \end{cases}$$

Finally, due to the periodicity, this dependence can be extended to arbitrary time as

$$r(NT+\tau) = r(\tau) \qquad N = 0, \pm 1, \pm 2 \tag{A11}$$

$$r(\tau) = r_1(|\tau|) \qquad \tau \in (-T/2, T/2). \tag{A12}$$

Note that the symmetry relation

$$r(-t) = r(t) \tag{A13}$$

follows from the relations (A11) and (A12).

The time dependence of ϕ is monotonic, because from (2.13) we have for its derivative

$$\dot{\phi}(t) = \frac{d\phi(t)}{dt} = \frac{L}{mr^2(t)} > 0 \tag{A14}$$

which can be integrated to give

$$\phi(t) = \frac{L}{m}\int_0^t \frac{dt'}{r^2(t')}. \tag{A15}$$

From this expression, using the property (A13), we conclude that $\phi(t)$ is an antisymmetric function:

$$\phi(-t) = -\phi(t) \tag{A16}$$

and using (A11) in (A15) for $\tau \in (-T/2, T/2)$, we find

$$\phi(NT+\tau) = N\Delta\phi + \phi(\tau) \qquad N = 0, \pm 1, \pm 2, \ldots. \tag{A17}$$

For the initial time interval ϕ is given by its first branch, (A3), so that

$$\phi(\tau') = \phi_1(r(\tau')) \qquad \tau' \in [0, T/2). \tag{A18}$$

The radial velocity $\dot{r} = dr/dt$ has the properties

$$\dot{r}(NT+\tau) = \dot{r}(\tau) \qquad N = 0, \pm 1, \pm 2, \ldots \tag{A19}$$

$$\dot{r}(-t) = -\dot{r}(t) \tag{A20}$$

which follow directly from (A11) and (A13). For the initial time interval, $\dot{r}$ is given by its first branch (A4), so that

$$\dot{r}(\tau') = \dot{r}_1(r(\tau')) \qquad \tau' \in [0, T/2). \tag{A21}$$

Equations (A19)–(A21), together with (A11)–(A13) and (A4) can be combined to give

$$\dot{r}(t) = \mathrm{sgn}(\dot{r}(t))\dot{r}_1(r(t)) \tag{A22a}$$

$$\mathrm{sgn}(\dot{r}(t)) = \begin{cases} +1 & \text{for } t = NT + \tau' \\ -1 & \text{for } t = NT - \tau' \end{cases} \tag{A22b}$$

where $\tau' \in [0, T/2)$, N integral.

In this way we have $\boldsymbol{r}$ against t and $\boldsymbol{p}$ against t by inserting into (2.9) and (2.11) the above expressions for time-dependent r ((A11)–(A13)), ϕ ((A16)–(A18)), $\dot{r}$ ((A19)–(A22)) and $\dot{\phi}$ (A14).

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